Maximum Likelihood Estimation for Lévy-driven Linear Parabolic SPDEs

(work in progress)

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Pathways into Mathematics of SPDEs:
A Workshop for Young Researchers

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UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

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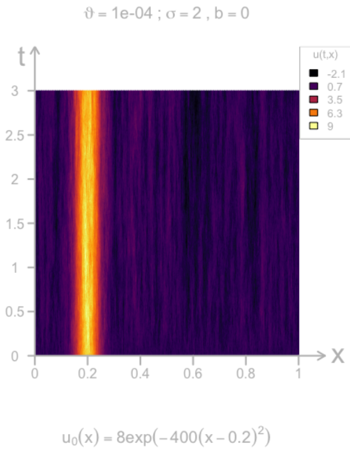
Outline

Motivation

Stochastic heat equation:



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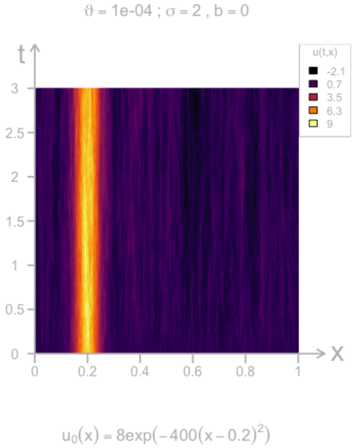

+ noise

with jumps

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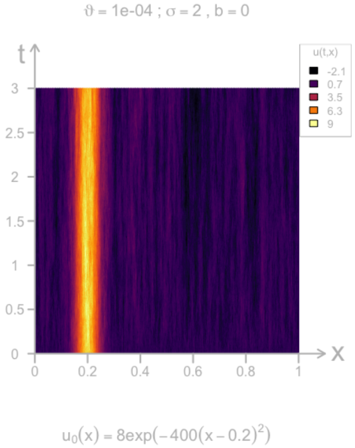

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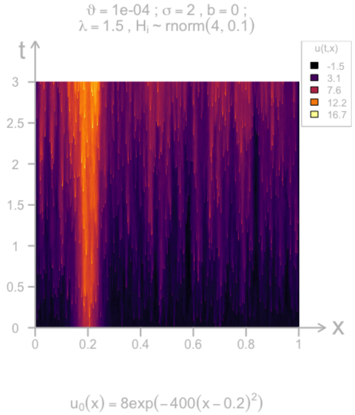

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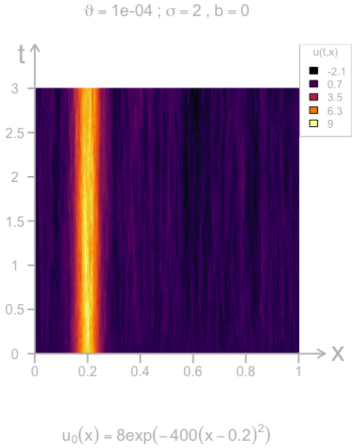

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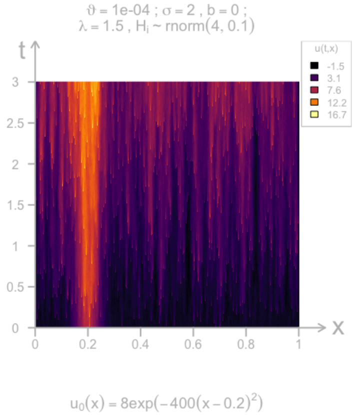

$\longrightarrow$ *Statistical consequences of jumps?*

Applications (examples)

- **Mathematical Finance**: model asset prices with jumps and heavy tails, portfolio optimisation under market shocks;
- **Physics**: systems with burst-like energy injections, phase transitions with rare and large fluctuations, water pollution detection;
- **Biology (Epidemiology)**: population dynamics with catastrophic events, spread of diseases with sudden outbreaks;
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Literature (excerpt)

- Q. Liang & J. Xiong & X. Zhao (2023). *Statistical estimation and nonlinear filtering in environmental pollution.*
- S. Peszat & J. Zabczyk (2007). *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach.*
- M. Huebner & B. Rozovskii (1995). *On asymptotic properties of maximum likelihood estimators for parabolic stochastic PDE's.*
- H. Mai (2014). *Efficient maximum likelihood estimation for Lévy-driven Ornstein—Uhlenbeck processes.*

2. Notation and setup

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A stochastic process $L := \big(L(t)\big)_{t\in[0,T]}$ is called a **Lévy process** iff

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$\rightsquigarrow$ Series expansion approach by [Peszat&Zabczyk, Sec. 4.8]:

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Assumption: $\forall n \in \mathbb{N}$: L_n are independent.

Solution concept and observation scheme (cts. time)

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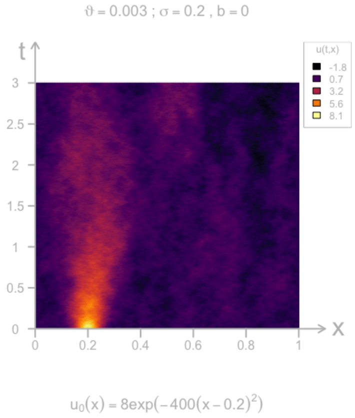
Time-continuous observation model:

For $t \in [0,T]$, we have access to a complete path $t \mapsto u^N(t,\cdot)$ on D.

Illustration

Stochastic heat equation (spectral approach):



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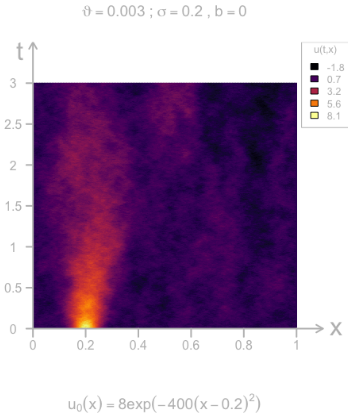

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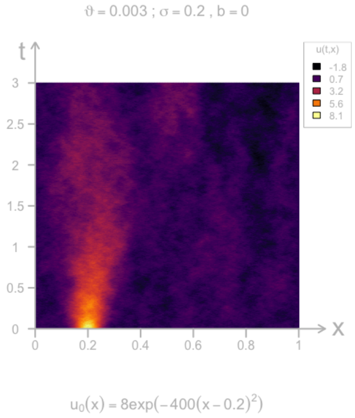

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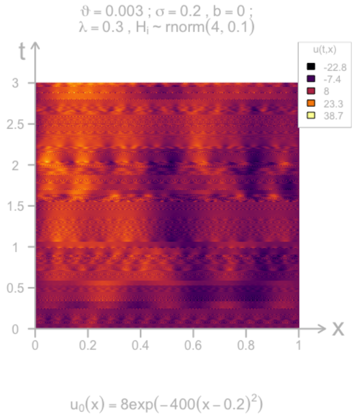
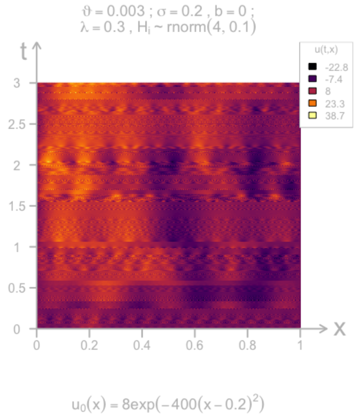


spectral approach ("mathematical model")



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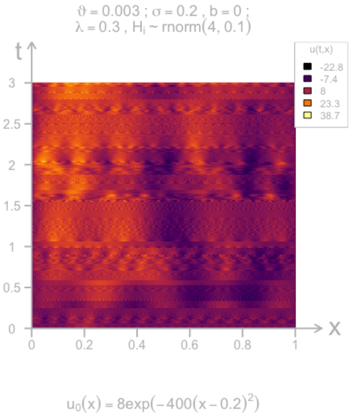

**spatial approach
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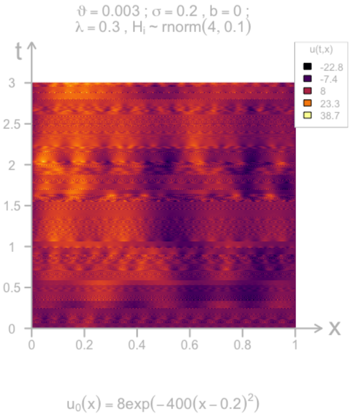

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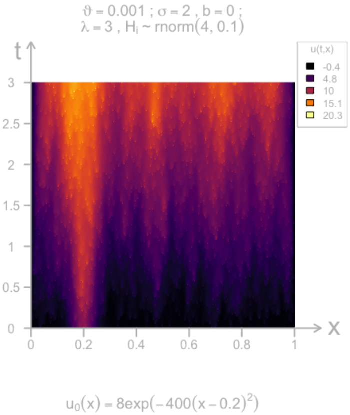

3. Maximum likelihood estimation via spectral approach

Maximum likelihood estimator (MLE): If $\sigma > 0$, then

$$\widehat{\vartheta}_N := \frac{\int_0^T \left\langle A\, u^N(t,\cdot),\, \mathrm{d}(u^N)^c(t,\cdot)\right\rangle_{L^2(D)}}{\int_0^T \|A\, u^N(t,\cdot)\|^2_{L^2(D)}\, \mathrm{d}t}, \quad \left(u^N\right)^c \text{ cts. martingale part under } \mathbb{P}^{N,0}.$$

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$\rightsquigarrow$ **Discretised MLE**: $\quad \overline{\vartheta}_{N,M} := \dfrac{\sum_{m=1}^M \langle A\, u^N(t_{m-1},\cdot),\, \Delta_m(u^N)^c(\cdot)\rangle_{L^2(D)}}{\sum_{m=1}^M \|A\, u^N(t_{m-1},\cdot)\|^2_{L^2(D)} \cdot \Delta_m}.$

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$\longrightarrow$ $f_{N,M} := N^{\alpha} \cdot M^{-\beta}$, for $\alpha \in \mathbb{R}$, $\beta \in (0, 1/2)$, with $f_{N,M} \to 0$.

Spectral jump filtering

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$\rightsquigarrow$ Make estimator feasible $\longrightarrow$ introduce jump filter to replace $\left(u^N\right)^c$!

Estimator with spectral jump filter:

$$\widehat{\vartheta}_{N,M} := \frac{\sum_{m=1}^{M} \left\langle A\, u^N(t_{m-1},\cdot),\ \boldsymbol{\Delta_m u^N(\cdot) \cdot \mathbb{1}\left\{ \left\|\Delta_m u^N\right\|_{L^2(D)} \leq f_{N,M} \right\}} \right\rangle_{L^2(D)}}{\sum_{m=1}^{M} \|A\, u^N(t_{m-1},\cdot)\|^2_{L^2(D)} \cdot \Delta_m}.$$

$\rightsquigarrow$ *How to choose the threshold sequence* $(f_{N,M})_{N,M\in\mathbb{N}}$*?*

$\longrightarrow f_{N,M} := N^{\alpha} \cdot M^{-\beta}$, for $\alpha \in \mathbb{R}$, $\beta \in (0, 1/2)$, with $f_{N,M} \to 0$.

$\rightsquigarrow$ Aim: Control

$$\begin{tikzcd} \boxed{\text{MLE } \widehat{\vartheta}_N} \arrow[r, "\textbf{discret. error}"] & \boxed{\text{discretised MLE } \overline{\vartheta}_{N,M}} \arrow[r, "\textbf{approx. error}"] & \boxed{\text{jump filter estimator } \widehat{\vartheta}_{N,M}} \end{tikzcd}$$

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Assumption

(i) Fourth moments of Lévy processes: $\forall\, n \in \mathbb{N}$: $\mathbb{E}\left[L_n(t)^4 \right] < \infty$.

(ii) Non-vanishing white noise intensity: $\sigma > 0$.

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Grant the above assumptions (i)–(v).

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Theorem (Altmeyer, TD, Strauch)

Grant the above assumptions (i)–(v). Then, if $c > 2p/d + 3$ *and* $p/d + 3/2 - c(1/2 - \beta) < \alpha < c\beta$, *under* $\mathbb{P}^{N,\vartheta_0}$, *it holds*

$$N^{\frac{d+2p}{2d}} \cdot \left(\widehat{\vartheta}_{N,M} - \vartheta_0\right) \xrightarrow{\mathrm{d}} \mathrm{N}(0, \vartheta_0 \cdot \Sigma) \quad \text{as } N, M \to \infty.$$

5. Conclusion

Summary

- Introduced stat. accessible setup (**spectral observations**) for **Lévy-driven linear parabolic SPDEs**;
- derived **ML-based estimators** and discussed their **feasibility** for **time-continuous** and **time-discrete observations**;
- found sufficient conditions for establishing a **spectral jump filter** into the SPDE setup;
- obtained **central limit theorems** that attain the **benchmark convergence rate** of the white noise setting; studied the **influence of (temporal) jumps** on the asymptotic variance.

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Outlook

- Establish **other estimation methods** for possibly unknown or entirely different parameters $\longrightarrow$ **non-parametric estimation**;
- consider **non-parabolic** or **non-linear SPDEs** with jumps;
- model **(correlated) spatial jumps** and examine their influence on statistics; study more **physical observation models** (local measurements) $\longrightarrow$ spatial asymptotics;
- **numerical studies** and **applications** to data.

Thank you!

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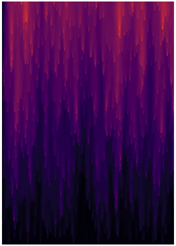

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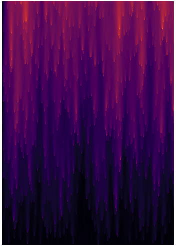 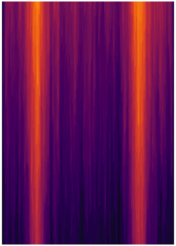

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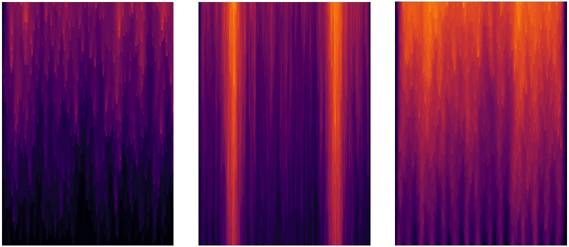

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